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# The third parafermionic chiral algebra with the symmetry $Z_3$

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Abstract.

We have constructed the parafermionic chiral algebra with the principal parafermionic fields $\Psi, \Psi^{+}$ having the conformal dimension $\Delta_{\Psi} = 8/3$ and realizing the symmetry $Z_3$.

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Most widely known is the first series of $Z_N$ parafermionic theories, which has been defined in [1]. Its numerous applications and appearances are well known. Less known is the second $Z_N$ series. Its chiral algebra has also been found in [1], while its representation theory has been defined rather recently, in [2]. For the particular case of $N = 3$, i.e. for the second $Z_3$ parafermionic algebra, the corresponding conformal field theory has been developed long ago, in [3].

Restricting our discussion to the $Z_3$ case, the first and the second $Z_3$ parafermions have respectively conformal dimension $\Delta_\Psi = 2/3$ and $\Delta_\Psi = 4/3$. More generally, the associativity constraints (a small fraction of them) allow for the following discrete set of values of $\Delta_\Psi$, for $Z_3$ parafermions:

$$\Delta_\Psi = \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{10}{3}, \text{etc.} = \frac{2l}{3}, \quad l = 1, 2, 4, 5, \text{etc.} \tag{1}$$

Classified in this order, in this paper we shall present the third solution of the associativity constraints: its chiral algebra, with the basic parafermions having the conformal dimension 8/3.

In this short communication we intend to present our main results on the third $Z_3$ parafermions. More details and demonstrations will be given in [4].

In the cases of the first and the second parafermions, the chiral algebra is made of $\Psi, \Psi^+$ and $T(z)$, the Virasoro part. The first main difference of the third $Z_3$ parafermions is that, in addition to the principal couple of chiral fields $\Psi(z), \Psi^+(z)$ with $\Delta_\Psi = \Delta_{\Psi^+} = 8/3$, the chiral algebra of local fields contains also the fields $\tilde{\Psi}(z), \tilde{\Psi}^+(z)$ with the dimension $\Delta_{\tilde{\Psi}} = \Delta_{\tilde{\Psi}^+} = \Delta_\Psi + 2$, and one boson field $B(z)$ with $\Delta_B = 4$, plus as usual $T(z)$.

This algebra is of the following form:

$$\Psi(z')\Psi(z) = \frac{1}{(z'-z)^{\Delta_\Psi}}\{\lambda\Psi^+(z) + (z'-z)\lambda\beta^{(1)}_{\Psi\Psi,\Psi^+}\partial\Psi^+(z)$$
$$+(z'-z)^2[\lambda\beta^{(11)}_{\Psi\Psi,\Psi^+}\partial^2\Psi^+(z) + \lambda\beta^{(2)}_{\Psi\Psi,\Psi^+}L_{-2}\Psi^+(z) + \zeta\tilde{\Psi}(z)] + ...\} \tag{2}$$

$$\Psi(z')\Psi^+(z) = \frac{1}{(z'-z)^{2\Delta_\Psi}}\{1 + (z'-z)^2\frac{2\Delta}{c}T(z) + (z'-z)^3\frac{\Delta}{c}\partial T(z)$$

$$+(z'-z)^4[\beta^{(112)}_{\Psi\Psi^+,I}\partial^2 T(z)+\beta^{(22)}_{\Psi\Psi^+,I}\Lambda(z)+\gamma B(z)]$$

$$+(z'-z)^5[\beta^{(1112)}\partial^3 T(z)+\beta^{(122)}\partial\Lambda(z)+\gamma\beta^{(1)}_{\Psi\Psi^+,B}\partial B(z)]+...\} \tag{3}$$

$$\Psi(z')\tilde{\Psi}(z)=\frac{1}{(z'-z)^{\Delta_\Psi+2}}\{\zeta\Psi^+(z)+(z'-z)^2[\zeta\beta^{(11)}_{\Psi\tilde{\Psi},\Psi^+}\partial^2\Psi^+(z)$$

$$+\zeta\beta^{(2)}_{\Psi\tilde{\Psi},\Psi^+}L_{-2}\Psi(z)+\eta\tilde{\Psi}^+(z)]+(z'-z)^3[\zeta\beta^{(111)}_{\Psi\tilde{\Psi},\Psi^+}\partial^3\Psi(z)+\zeta\beta^{(12)}_{\Psi\tilde{\Psi},\Psi^+}\partial L_{-2}\Psi(z)$$

$$+\zeta\beta^{(3)}_{\Psi\tilde{\Psi},\Psi^+}L_{-3}\Psi(z)+\eta\beta^{(1)}_{\Psi\tilde{\Psi},\tilde{\Psi}^+}\partial\tilde{\Psi}^+(z)]+...\} \tag{4}$$

$$\Psi(z')\tilde{\Psi}^+(z)=\frac{1}{(z'-z)^{2\Delta_\Psi-2}}\{\mu B(z)+(z'-z)\mu\beta^{(1)}_{\Psi\tilde{\Psi}^+,B}\partial B(z)+...\} \tag{5}$$

$$\Psi(z')B(z)=\frac{1}{(z'-z)^4}\{\gamma\Psi(z)+(z'-z)\gamma\beta^{(1)}_{\Psi B,\Psi}\partial\Psi(z)+(z'-z)^2[\gamma\beta^{(11)}_{\Psi B,\Psi}\partial^2\Psi(z)$$

$$+\gamma\beta^{(2)}_{\Psi B,\Psi}L_{-2}\Psi(z)+\mu\tilde{\Psi}(z)]+(z'-z)^3[\gamma\beta^{(111)}_{\Psi B,\Psi}\partial^3\Psi(z)$$

$$+\gamma\beta^{(12)}_{\Psi B,\Psi}\partial L_{-2}\Psi(z)+\gamma\beta^{(3)}_{\Psi B,\Psi}L_{-3}\Psi(z)+\mu\beta^{(1)}_{\Psi B,\Psi}\partial\tilde{\Psi}(z)]+...\}, \tag{6}$$

where the $L_n$ represent the modes of $T(z)$ and the coefficients $\beta^{\cdot\cdot}_{\cdot\cdot}$ are fixed by the conformal symmetry.

Besides this principal products, which are actually sufficient to address the calculus of representations, one must consider also the following expansions:

- $\tilde{\Psi}\times\tilde{\Psi}\sim\eta[\Psi]+\tilde{\lambda}[\tilde{\Psi}]$. Here and in the following, the notation $[A]$ indicates the operator $A$ and its Virasoro descendants.

- $\tilde{\Psi}\times\tilde{\Psi}^+\sim[I]+\tilde{\gamma}[B(z)]$, with $I$ the identity operator.

- $\tilde{\Psi}\times B\sim\mu[\Psi]+\tilde{\gamma}[\tilde{\Psi}]$.

- $B\times B\sim[I]+\xi[B]$.

The Virasoro content of these developments is standard, containing the (Virasoro) descendants of the above mentioned chiral fields with the corresponding $\beta$ coefficients. Two rules have to be respected everywhere: these developments should go (i.e. made explicit in terms of the above mentioned fields) only up to the operators of the fifth level in the $Z_3$

neutral sector (which corresponds to the number of singular terms in (3)), and up to the operators of the third level in the $Z_3$ charged sector. In (2), the third level operators are not shown because in the case of the symmetric product $(\Psi\Psi)$ the third level terms are uniquely defined in terms of the derivatives of the preceding levels operators; they do not provide new relations in the eventual calculations of the highest weight representations. They could be added in (2), but in this case they are irrelevant.

One could verify that, with the above prescriptions concerning the explicit terms in the equations (2)-(6), the corresponding representation theory is well defined: one could define all the matrix elements needed in the degeneracy calculations [4].

There are a total of 8 coupling constants entering into the equations (2)-(6) and into the four expansions listed above. They could be defined through the following three-point functions:

$$
\begin{aligned}
&< \Psi\Psi\Psi >= \lambda, \quad < \Psi\Psi\tilde{\Psi} >= \zeta, \quad < \Psi\Psi^{+}B >= \gamma \\
&< \Psi\tilde{\Psi}\tilde{\Psi} >= \eta, \quad < \tilde{\Psi}\tilde{\Psi}\tilde{\Psi} >= \tilde{\lambda}, \quad < \Psi\tilde{\Psi}^{+}B >= \mu \\
&< \tilde{\Psi}\tilde{\Psi}^{+}B >= \tilde{\gamma}, \quad < BBB >= \xi
\end{aligned}
\tag{7}
$$

Here $< \Psi\Psi\Psi >=< \Psi(\infty)\Psi(1)\Psi(0) >$ etc. .

The associativity constraints fix these constants in terms of the Virasoro algebra central charge $c$ which remains a free parameter. The results are given below:

$$
\lambda = \frac{14}{27}\sqrt{3}\sqrt{\frac{c+32}{c}} \tag{8}
$$

$$
\zeta = \frac{8}{27}\sqrt{30}\sqrt{\frac{(c+56)(11c+14)}{c(784+57c)}} \tag{9}
$$

$$
\gamma = \frac{4}{27}\sqrt{15}\sqrt{\frac{(c+56)(11c+14)}{c(22+5c)}} \tag{10}
$$

$$
\eta = \frac{7}{27}\frac{(349c+2688)\sqrt{3}}{784+57c}\sqrt{\frac{c+32}{c}} \tag{11}
$$

$$
\tilde{\lambda} = \frac{98}{135}\sqrt{30}\frac{(c+32)(4877c^2+51466c+13104)}{\sqrt{c(c+56)(11c+14)(784+57c)^3}} \tag{12}
$$

$$\mu = \frac{28}{9}\sqrt{6}\sqrt{\frac{(c+32)(22+5c)}{c(784+57c)}} \tag{13}$$

$$\tilde{\gamma} = \frac{7}{135}\sqrt{15}\frac{(20595c^3+823534c^2+5532912c+2121728)}{\sqrt{c(22+5c)(c+56)(11c+14)(784+57c)}} \tag{14}$$

$$\xi = \frac{2}{15}\sqrt{15}\frac{(85c^2+2566c+976)}{\sqrt{c(22+5c)(c+56)(11c+14)}} \tag{15}$$

Some details on the techniques used to obtain theses results will be given in [4].

It could be observed that for $c = -14/11$ this $Z_3$ algebra closes by just $\Psi, \Psi^+$ and $T$, while the fields $\tilde{\Psi}, \tilde{\Psi}^+, B$ decouple from the theory. At this particular value of $c$, the fields $\Psi, \Psi^+$ belong to the Virasoro non-unitary minimal model with $p'/p = 11/6$, $\Psi, \Psi^+ \sim \Phi_{1,3}$.

Certain analogies with the $Z_2$ symmetric chiral algebras could also be mentioned here. The allowed dimensions for the $\Psi$ operators, which realize the $Z_2$ multiplication rules (and which are fermionic in this case) are the following:

$$\Delta_\Psi = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2} ... \tag{16}$$

Again, classified in this way by the sequence of the dimensions $\Delta_\Psi$, the first solution for the associative chiral algebra with $Z_2$ symmetry corresponds to free fermions (and the Ising model). The second solution, with $\Delta_\Psi = \frac{3}{2}$, corresponds to the $N = 1$ superconformal algebra, which, in a sense, is just a less trivial realization of the $Z_2$ symmetry, as compared to free fermions.

The third solution, with $\Delta_\Psi = 5/2$, is know as the chiral algebra of the $WB_2$ model [5]. Starting with the third solution, the associativity of the chiral algebra of the $\Psi$ field and its closure (while preserving $c$ as a free parameter) requires the introduction of extra fields: $B(z)$ chiral field with $\Delta_B = 4$ in case of $\Delta_\Psi = 5/2$; more bosonic fields for still higher solutions in the sequence (16) (they are the models $WB_n$ [5]).

In general, when the value of $\Delta_\Psi$ grows, the number of singular terms in the expansions of $\Psi(z')\Psi(z)$ and $\Psi(z')\Psi^+(z)$, which have to be defined explicitly, grows also. The gap becomes too big to be filled by just Virasoro descendants of $\Psi, \Psi^+$ and to satisfy associativity.

The appearance of extra conserved currents for higher solutions has also a physical meaning. Indeed, for a given symmetry, higher solutions are likely to correspond to higher multicriticalities. Naturally, the number of degrees of freedom of the corresponding (multi)critical theories should grow.

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